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Airy Quantization for Half-Line Schrödinger Operators with Mixed Boundary Conditions and Perturbed Linear Potentials

Cuantización de Airy para operadores de Schrödinger en la semirrecta con condiciones de frontera mixtas y potenciales lineales perturbados

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RESUMEN

En este trabajo se desarrolla un marco espectral autoadjunto para operadores de Schrödinger definidos en la semirrecta con potencial lineal y condiciones de frontera mixtas dentro del contexto de las funciones de Airy. Se construye la realización autoadjunta correspondiente y se demuestra que es semidefinida inferiormente y posee resolvente compacto, por lo que su espectro es puramente discreto. En este marco se obtiene una ley de cuantización unificada para las condiciones de frontera de Dirichlet, Neumann y Robin, y se establecen la simplicidad del espectro mixto, la ortogonalidad de las funciones propias, el entrelazamiento de las raíces positivas y fórmulas cerradas de normalización. Se obtiene, además, la asintótica de Weyl de los autovalores mixtos positivos y se identifica el parámetro de Robin crítico $\rho_c = -\mathrm{Ai}'(0)/\mathrm{Ai}(0)$, que separa un espectro estrictamente positivo de la aparición de exactamente un

ABSTRACT

This paper develops a self-adjoint spectral framework for half-line Schrödinger operators with a linear potential and mixed boundary conditions in the Airy setting. We construct the corresponding self-adjoint realization and prove that it is bounded from below and has compact resolvent, implying a purely discrete spectrum. Within this framework, we derive a unified Airy quantization law for Dirichlet, Neumann, and Robin boundary conditions, and establish the simplicity of the mixed spectrum, orthogonality of eigenfunctions, interlacing of the positive roots, and closed normalization formulas. We also obtain the Weyl asymptotics of the positive mixed eigenvalues and identify the critical Robin parameter $\rho_c = -\mathrm{Ai}'(0)/\mathrm{Ai}(0)$*, which separates a strictly positive spectrum from the appearance of exactly one negative eigenvalue. We further analyze the dependence of the spectrum and the normalized eigen-*

autovalor negativo. Asimismo, se analiza la influencia de la condición de frontera sobre la distribución espectral y la localización espacial de las funciones propias normalizadas. Para operadores perturbados de la forma $Fx + \varepsilon W(x)$, con $W \in L^\infty(0,\infty)$, se obtienen expresiones explícitas de primer orden para las correcciones de los autovalores y las funciones propias mediante integrales que involucran funciones de Airy. Como aplicaciones ilustrativas se consideran perturbaciones acotadas con decaimiento exponencial y racional; los valores numéricos para $\rho = 0$ y $\rho = 0.5$ muestran que la perturbación racional produce correcciones mayores. El análisis muestra que la interacción de frontera controla por sí sola los niveles de energía y la localización espacial de los estados ligados, mientras que el comportamiento confinante del potencial lineal preserva la discreción del espectro bajo perturbaciones acotadas. En conjunto, los resultados proporcionan un marco analítico unificado que conecta la teoría espectral de operadores autoadjuntos, las funciones especiales de Airy y los modelos cuánticos unidimensionales con interacciones de frontera.

Palabras clave: funciones de Airy; condiciones de frontera mixtas; operadores autoadjuntos; teoría espectral; potenciales lineales perturbados

functions on the boundary condition, providing a spectral interpretation of the transition between Dirichlet, Neumann, and Robin realizations. For perturbed operators of the form $Fx + \varepsilon W(x)$, with $W \in L^\infty(0,\infty)$, we obtain explicit first-order corrections to eigenvalues and eigenfunctions in terms of Airy-function integrals, illustrating the theory with exponentially and rationally decaying bounded perturbations. Numerical values for $\rho = 0$ and $\rho = 0.5$ show that the rational perturbation produces larger first-order shifts and that the response to the Robin parameter depends on the state. The analysis shows that the boundary interaction alone controls the energy levels and the spatial localization of the bound states, while the dominant confining behavior of the linear potential preserves the discreteness of the spectrum under bounded perturbations. The results provide a unified analytical framework connecting self-adjoint spectral theory, Airy special functions, and one-dimensional quantum models with boundary interactions.

Keywords: *Airy functions; mixed boundary conditions; self-adjoint operators; spectral theory; perturbed linear potentials*

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1. INTRODUCTION

Airy functions play a fundamental role in asymptotic analysis, spectral theory, and mathematical physics, providing the canonical local model for the transition between oscillatory and exponential regimes [1, 2, 6, 9]. In one-dimensional quantum mechanics, the reduction of Schrödinger's equation with a linear potential to the Airy equation is classical [3, Ch. II]. Related structures also arise in triangular quantum wells and band-bending models [4, Ch. IV], as well as in the quantum bouncer, where the spectrum is determined by the zeros of Ai [5]. These examples illustrate that Airy functions provide a natural analytical framework for boundary selection and quantization in linear-potential problems.

The subject continues to attract considerable attention. Recent developments include higher-order Airy-type equations and related pseudo-processes [17], generalized Airy polynomials and Hankel determinants [18], and renewed analytical studies of classical Airy theory [19]. From the operator-theoretic viewpoint, recent work on half-line Stark operators has clarified the effects of mixed boundary conditions and perturbations on spectral data [14], while standard monographs continue to provide the functional-analytic framework for self-adjoint realizations and compact resolvents [11, 12, 13, 16]. Related developments involving classical orthogonal polynomial families include differential and

electrostatic analyses of Jacobi–Sobolev polynomials, together with Jacobi–spectral formulations of boundary-value problems based on weighted Sturm–Liouville realizations and transform diagonalization [7, 15].

A broader Airy-based perspective connecting beam models and quantum systems with linear potentials was developed in [10]. In contrast, the present paper focuses on a specific half-line spectral problem with mixed boundary conditions and bounded perturbations, developing its self-adjoint realization, an explicit Airy quantization law, and a first-order perturbative analysis. Related applications in quantum mechanics involving spectral representations of Hamiltonian operators and classical orthogonal polynomials can also be found in [20, 21].

More precisely, we consider the half-line problem

$$-\frac{\hbar^2}{2m}\psi''(x) + Fx\,\psi(x) = E\psi(x), \qquad x > 0, \tag{1}$$

subject to the boundary condition

$$a\psi(0) + b\psi'(0) = 0, \qquad (a, b) \neq (0, 0),$$

and show that the corresponding self-adjoint realization is bounded from below and has compact resolvent. Consequently, the spectrum is purely discrete. Within this framework, we derive a unified Airy quantization law encompassing the Dirichlet, Neumann, and Robin cases, and establish the simplicity of the mixed spectrum, the orthogonality of eigenfunctions, the interlacing of the positive roots, and closed normalization formulas.

We also study bounded perturbations of the form

$$H_\varepsilon = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + Fx + \varepsilon W(x), \qquad W \in L^\infty(0, \infty),$$

subject to the same mixed boundary condition. Under these assumptions, we obtain first-order corrections to the eigenvalues and eigenfunctions in terms of explicit integrals involving Airy functions, and illustrate the perturbative approach with exponentially and rationally decaying bounded potentials.

The contribution of this paper is twofold. First, it places the half-line Airy problem with mixed boundary conditions within a rigorous self-adjoint spectral framework. Second, it shows that the same Airy structure remains effective under bounded perturbations of the linear potential. Rather than presenting a survey of Airy theory, the paper provides a self-contained spectral treatment of the half-line linear-potential problem with mixed boundary conditions and bounded perturbations.

The paper is organized as follows. Section 3.1 develops the self-adjoint realization of the half-line operator and its basic spectral properties. Section 3.2 presents the Airy reduction on the half-line. Section 3.3 establishes the unified quantization law under mixed boundary conditions and derives the main spectral consequences. Section 3.4 studies perturbed linear potentials and derives the first-order corrections. The final sections discuss the significance of the results and summarize the main conclusions.

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2. PRELIMINARIES

This section collects the functional-analytic and differential-equation tools used throughout the paper. Airy-specific material—including power-series constructions, Bessel-type reductions, contour-integral representations, and the zero structure of Ai and Bi—was developed in [10]. Accordingly, we recall only those classical results needed for the self-adjoint realization of the half-line problem, the proof of orthogonality, and the spectral analysis of mixed boundary conditions; see also [11, 16, 14, 13].

Throughout the paper, $L^2(0,\infty)$ denotes the complex Hilbert space $L^2(0,\infty;\mathbb{C})$, equipped with the standard sesquilinear inner product

$$\langle u, v\rangle = \int_0^\infty u(x)\overline{v(x)}\,\mathrm{d}x.$$

2.1 Endpoint traces on the half-line

For $u \in H^1(0,\infty) \equiv W^{1,2}(0,\infty) = \{u \in L^2(0,\infty) : u' \in L^2(0,\infty)\}$, the trace $u(0)$ is well defined, and the trace operator $\gamma_0 : H^1(0,\infty) \to \mathbb{C}$, $\gamma_0 u = u(0)$, is continuous. A quantitative version of this result is provided by the following one-dimensional trace inequality.

Lemma 1 (Trace inequality on the half-line)**.** *For every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$|u(0)|^2 \le \varepsilon\|u'\|^2_{L^2(0,\infty)} + C_\varepsilon\|u\|^2_{L^2(0,\infty)}, \qquad u \in H^1(0,\infty). \tag{2}$$

Proof. Let $\delta > 0$. For every $x \in (0,\delta)$, the Fundamental Theorem of Calculus gives

$$u(0) = u(x) - \int_0^x u'(t)\,\mathrm{d}t.$$

Hence, by the Cauchy–Schwarz inequality,

$$|u(0)|^2 \le 2|u(x)|^2 + 2x\int_0^x |u'(t)|^2\,\mathrm{d}t.$$

Integrating with respect to $x \in (0,\delta)$ and using $x \le \delta$, we obtain

$$\delta|u(0)|^2 \le 2\int_0^\delta |u(x)|^2\,\mathrm{d}x + 2\delta^2\int_0^\delta |u'(x)|^2\,\mathrm{d}x.$$

Therefore,

$$|u(0)|^2 \le \frac{2}{\delta}\|u\|^2_{L^2(0,\infty)} + 2\delta\|u'\|^2_{L^2(0,\infty)}.$$

Choosing $\delta = \varepsilon/2$, we obtain

$$|u(0)|^2 \le \varepsilon\|u'\|^2_{L^2(0,\infty)} + \frac{4}{\varepsilon}\|u\|^2_{L^2(0,\infty)}.$$

Thus the result follows with $C_\varepsilon = 4/\varepsilon$. □

Remark 1. Lemma 1 provides the basic estimate needed to treat the Robin boundary term in the quadratic-form realization of the half-line operator.

2.2 Closed semibounded quadratic forms

Let $\mathcal{H}$ be a complex Hilbert space. A sesquilinear form $q : Q(q) \times Q(q) \to \mathbb{C}$ with dense domain $Q(q) \subset \mathcal{H}$ is called *semibounded* if there exists a constant $c \in \mathbb{R}$ such that $q[u] := q(u,u) \geq c\|u\|^2$ for every $u \in Q(q)$. Fix $M \geq -c$. The associated form norm is defined by

$$\|u\|_q := \left(q[u] + (M+1)\|u\|^2\right)^{1/2}, \qquad u \in Q(q). \tag{3}$$

The form q is said to be *closed* if $Q(q)$ is complete with respect to $\|\cdot\|_q$.

The following representation theorem is fundamental.

Theorem 1 (Representation theorem for closed semibounded forms)**.** *Let q be a densely defined, symmetric, closed, semibounded form on $\mathcal{H}$. Then there exists a unique self-adjoint operator A on $\mathcal{H}$, bounded from below, such that*

$$q(u,v) = \langle u, Av\rangle, \qquad u \in Q(q), \quad v \in D(A),$$

and $D(A) \subset Q(q)$.

Remark 2. Theorem 1 provides the fundamental tool for passing from the variational formulation of the half-line Stark operator to its self-adjoint realization with mixed boundary conditions.

2.3 Compact embeddings and compact resolvent

Let A be a closed operator on a Hilbert space $\mathcal{H}$. Its graph norm is defined by

$$\|u\|_{D(A)} := \left(\|u\|^2 + \|Au\|^2\right)^{1/2}, \qquad u \in D(A).$$

Definition 1. *Let A be a closed operator on $\mathcal{H}$ with nonempty resolvent set $\rho(A)$. We say that A has* a compact resolvent *if, for some (equivalently, every) $\lambda \in \rho(A)$, the operator $(A - \lambda I)^{-1} : \mathcal{H} \to \mathcal{H}$ is compact.*

The following criterion will be used repeatedly throughout the paper.

Theorem 2 (Compact resolvent criterion)**.** *Let A be a closed operator on $\mathcal{H}$ with $\rho(A) \neq \varnothing$. Then the following statements are equivalent:*

1. *A has compact resolvent.*
2. *$D(A)$, equipped with the graph norm, is compactly embedded in $\mathcal{H}$.*

In the quadratic-form setting, the following consequence is often useful.

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Corollary 1. *Let q be a densely defined, closed, semibounded form on $\mathcal{H}$, and let A denote the self-adjoint operator associated with Theorem 1. If the form domain $Q(q)$ is compactly embedded in $\mathcal{H}$, then A has compact resolvent.*

Remark 3. Corollary 1 provides the key mechanism for establishing the discreteness of the spectrum of the half-line Stark operator with confining potential Fx.

2.4 Self-adjoint operators with compact resolvent

The following classical result describes the spectral consequences of compact resolvent.

Theorem 3 (Discrete spectral theorem)**.** *Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$, bounded from below, and suppose that A has compact resolvent. Then the spectrum of A is purely discrete. More precisely, there exists a sequence of real eigenvalues*

$$\lambda_1 \leq \lambda_2 \leq \cdots \nearrow +\infty,$$

counted according to multiplicity, and an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of A.

As an immediate consequence, eigenfunctions corresponding to distinct eigenvalues are orthogonal.

Corollary 2. *If $Au_n = \lambda_n u_n$ and $Au_m = \lambda_m u_m$ with $\lambda_n \neq \lambda_m$, then $\langle u_n, u_m \rangle = 0$.*

2.5 Lagrange identity for second-order differential expressions

Let $\tau u := -p(x)u'' + q(x)u$, where $p > 0$ is constant and q is real-valued. If u and v are sufficiently smooth, then

$$(\tau u)\,\overline{v} - u\,\overline{\tau v} = -\frac{\mathrm{d}}{\mathrm{d}x}\Big(p\,u'\overline{v} - p\,u\,\overline{v'}\Big). \tag{4}$$

Integrating (4) over the interval (α, β) yields the Lagrange identity

$$\int_\alpha^\beta \big((\tau u)\,\overline{v} - u\,\overline{\tau v}\big)\,\mathrm{d}x = -\Big[p\,u'\overline{v} - p\,u\,\overline{v'}\Big]_\alpha^\beta. \tag{5}$$

Remark 4. Identity (5) is the one-dimensional Green's identity associated with the differential expression τ. It will be used later to prove the orthogonality of eigenfunctions satisfying the same boundary condition.

2.6 Simplicity of zeros for second-order linear equations

The following elementary uniqueness result will also be used.

Lemma 2 (Simplicity of zeros)**.** *Let y be a solution of the second-order linear ordinary differential equation*

$$y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0,$$

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on an interval I, where a_0 and a_1 are continuous. If there exists $x_0 \in I$ such that

$$y(x_0) = 0, \qquad y'(x_0) = 0,$$

then $y \equiv 0$ on I. In particular, every zero of a nontrivial solution is simple.

Proof. The result follows immediately from the uniqueness theorem for the Cauchy problem associated with linear ordinary differential equations with continuous coefficients. □

3. RESULTS

3.1 Self-adjoint realization and basic spectral properties

Let

$$c := \frac{\hbar^2}{2m}, \qquad \tau u := -c\,u'' + F x\,u, \qquad F > 0.$$

To formulate the half-line problem within an operator-theoretic framework, we introduce the energy space

$$\mathcal{Q} := \Big\{ u \in H^1(0,\infty) : \ \sqrt{x}\,u \in L^2(0,\infty) \Big\},$$

endowed with the norm

$$\|u\|_{\mathcal{Q}}^2 := \|u\|_{L^2(0,\infty)}^2 + \|u'\|_{L^2(0,\infty)}^2 + \|\sqrt{x}\,u\|_{L^2(0,\infty)}^2.$$

Lemma 3 (Compact embedding of the energy space)**.** *The embedding*

$$\mathcal{Q} \hookrightarrow L^2(0,\infty)$$

is compact.

Proof. Let $\{u_n\}$ be a bounded sequence in $\mathcal{Q}$. Then there exists $M > 0$ such that $\|u_n\|_{L^2} + \|u_n'\|_{L^2} + \|\sqrt{x}\,u_n\|_{L^2} \le M$ for all n. Since $\mathcal{Q}$ is a Hilbert space, there exist a subsequence, still denoted by $\{u_n\}$, and a function $u \in \mathcal{Q}$ such that $u_n \rightharpoonup u$ weakly in $\mathcal{Q}$. Hence, for every fixed $R > 0$, we also have $u_n \rightharpoonup u$ weakly in $H^1(0,R)$, and the Rellich–Kondrachov theorem yields $u_n \to u$ strongly in $L^2(0,R)$.

Furthermore, since $x \ge R$ on (R,∞),

$$\|u_n\|_{L^2(R,\infty)}^2 \le R^{-1}\|\sqrt{x}\,u_n\|_{L^2}^2 \le \frac{M^2}{R}.$$

By weak lower semicontinuity of the norm, $\|\sqrt{x}\,u\|_{L^2} \le M$, and therefore $\|u\|_{L^2(R,\infty)} \le M/\sqrt{R}$. Consequently, $\|u_n - u\|_{L^2(R,\infty)} \le 2M/\sqrt{R}$.

Given $\varepsilon > 0$, choose $R > 0$ such that $\frac{2M}{\sqrt{R}} < \frac{\varepsilon}{2}$, and then choose n sufficiently large so that $\|u_n - u\|_{L^2(0,R)} < \varepsilon/2$. Therefore, $\|u_n - u\|_{L^2(0,\infty)} < \varepsilon$.

Hence every bounded sequence in $\mathcal{Q}$ admits a subsequence converging strongly in $L^2(0,\infty)$. Therefore, the embedding $\mathcal{Q} \hookrightarrow L^2(0,\infty)$ is compact. □

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Proposition 1 (Self-adjoint realization)**.** *Let $a, b \in \mathbb{R}$, with $(a, b) \neq (0, 0)$. Consider the differential expression $\tau u = -c\,u'' + Fx\,u$ on $(0, \infty)$, subject to the boundary condition*

$$a\,u(0) + b\,u'(0) = 0. \tag{6}$$

Then there exists a self-adjoint operator $H_{a,b}$ in $L^2(0, \infty)$ associated with τ and the boundary condition (6). Moreover, $H_{a,b}$ is bounded from below and has compact resolvent. Consequently, its spectrum is purely discrete and consists of real eigenvalues of finite multiplicity that accumulate only at $+\infty$.

Proof. We distinguish between the Dirichlet case $b = 0$ and the Robin case $b \neq 0$.

Case 1: $b = 0$ **(Dirichlet).** In this case, the boundary condition (6) reduces to $u(0) = 0$. Define $\mathcal{Q}_D := \{u \in \mathcal{Q} : u(0) = 0\}$ and the quadratic form

$$q_D[u, v] := c \int_0^\infty u'(x)\overline{v'(x)}\,\mathrm{d}x + F \int_0^\infty x\,u(x)\overline{v(x)}\,\mathrm{d}x, \qquad u, v \in \mathcal{Q}_D.$$

The form q_D is symmetric and nonnegative. It is densely defined because $C_c^\infty(0, \infty) \subset \mathcal{Q}_D$ and $C_c^\infty(0, \infty)$ is dense in $L^2(0, \infty)$. Moreover, the continuity of the trace operator $\gamma_0 : H^1(0, \infty) \to \mathbb{C}$, defined by $\gamma_0 u = u(0)$, implies that $\mathcal{Q}_D$ is a closed subspace of $\mathcal{Q}$.

Since $q_D \geq 0$, we may choose $M = 0$ in the form norm (3). Hence,

$$\left(q_D[u] + \|u\|^2_{L^2(0,\infty)}\right)^{1/2}$$

coincides with the form norm on $\mathcal{Q}_D$. Since $\mathcal{Q}_D$ is a closed subspace of the Hilbert space $\mathcal{Q}$, it is complete with respect to this norm. Therefore, q_D is closed.

By Theorem 1, there exists a unique self-adjoint operator H_D, bounded from below, associated with q_D. This operator is the self-adjoint realization of $\tau u = -c\,u'' + Fx\,u$ subject to the Dirichlet boundary condition $u(0) = 0$.

Case 2: $b \neq 0$ **(Robin).** Set

$$\beta := -\frac{a}{b},$$

so that (6) becomes

$$u'(0) = \beta\,u(0). \tag{7}$$

Define the sesquilinear form on $\mathcal{Q}$ by

$$q_\beta[u, v] := c \int_0^\infty u'(x)\overline{v'(x)}\,\mathrm{d}x + F \int_0^\infty x\,u(x)\overline{v(x)}\,\mathrm{d}x + c\beta\,u(0)\overline{v(0)}, \qquad u, v \in \mathcal{Q},$$

and let

$$q_0[u, v] := c \int_0^\infty u'(x)\overline{v'(x)}\,\mathrm{d}x + F \int_0^\infty x\,u(x)\overline{v(x)}\,\mathrm{d}x.$$

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By Lemma 1, for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|u(0)|^2 \leq \varepsilon \|u'\|^2_{L^2(0,\infty)} + C_\varepsilon \|u\|^2_{L^2(0,\infty)}, \qquad u \in H^1(0,\infty).$$

Consequently,

$$|c\beta|\,|u(0)|^2 \leq c|\beta|\varepsilon \|u'\|^2_{L^2(0,\infty)} + c|\beta| C_\varepsilon \|u\|^2_{L^2(0,\infty)}.$$

Choosing $\varepsilon > 0$ sufficiently small, the boundary term is q_0-bounded with relative bound strictly smaller than one. Therefore, the KLMN theorem [12](Thm. X.17) implies that q_β is closed and semibounded on $\mathcal{Q}$. Since the boundary contribution is symmetric and q_0 is densely defined, q_β is also symmetric and densely defined. Moreover, since q_β is semibounded, its associated form norm is precisely the norm introduced in (3). Hence $\mathcal{Q}$, being a Hilbert space, is complete with respect to this norm, and therefore q_β is closed.

By Theorem 1, there exists a unique self-adjoint operator H_β, bounded from below, associated with q_β.

It remains to identify the boundary condition associated with the form. Let $u, v \in C_c^\infty([0,\infty))$. Integration by parts gives

$$\begin{aligned} q_\beta[u,v] &= c\int_0^\infty u'(x)\overline{v'(x)}\,\mathrm{d}x + F\int_0^\infty x\,u(x)\overline{v(x)}\,\mathrm{d}x + c\beta u(0)\overline{v(0)} \\ &= \int_0^\infty \Big(-c\,u''(x) + Fx\,u(x)\Big)\overline{v(x)}\,\mathrm{d}x + c\Big(\beta u(0) - u'(0)\Big)\overline{v(0)}. \end{aligned}$$

Hence the boundary contribution vanishes precisely when $u'(0) = \beta u(0)$, which is equivalent to the original boundary condition $a\,u(0) + b\,u'(0) = 0$. Thus, the self-adjoint operator determined by q_β is the realization of τ subject to the Robin boundary condition (7).

In both the Dirichlet and Robin cases, the form domain is compactly embedded in $L^2(0,\infty)$ by Lemma 3. Hence, Corollary 1 implies that the associated operator has compact resolvent. The conclusion then follows from Theorem 3. □

Remark 5. The sesquilinear form q_β is the canonical variational form associated with the differential expression $\tau u = -c\,u'' + Fx\,u$ subject to the Robin boundary condition $u'(0) = \beta u(0)$. The preceding proof shows that this boundary condition is precisely the one encoded by the variational formulation. The passage from the closed semibounded form q_β to the corresponding self-adjoint operator follows from the representation theorem for quadratic forms; see, for example, [8, 12, 16].

3.2 Airy reduction on the half-line

We consider the stationary Schrödinger equation (1) on the half-line, where $m > 0$ and $F > 0$. We impose square integrability at $+\infty$ together with the mixed boundary condition at $x = 0$, introduced in Section 3.3.

We introduce the characteristic length and the dimensionless spectral parameter

$$\ell := \left(\frac{\hbar^2}{2mF}\right)^{1/3}, \qquad \lambda := \frac{E}{F\ell}, \qquad \zeta := \frac{x}{\ell} - \lambda. \tag{8}$$

Using $\frac{\mathrm{d}}{\mathrm{d}x} = \frac{1}{\ell}\frac{\mathrm{d}}{\mathrm{d}\zeta}$ and $\frac{\mathrm{d}^2}{\mathrm{d}x^2} = \frac{1}{\ell^2}\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}$, equation (1) reduces to $\psi_{\zeta\zeta} = \zeta\psi$, namely, the classical Airy equation. Its general solution is

$$\psi(x) = A\mathrm{Ai}\left(\frac{x}{\ell} - \lambda\right) + B\mathrm{Bi}\left(\frac{x}{\ell} - \lambda\right).$$

Since $\mathrm{Ai}(\zeta) \to 0$ and $\mathrm{Bi}(\zeta) \to +\infty$ as $\zeta \to +\infty$, square integrability requires that $B = 0$. Therefore, every admissible eigenfunction is of the form

$$\psi(x) = C\mathrm{Ai}\left(\frac{x}{\ell} - \lambda\right). \tag{9}$$

The spectral problem is therefore reduced to determining those values of λ for which (9) satisfies the boundary condition at the origin.

Remark 6. Although this reduction is classical, in the present setting it must be interpreted together with Proposition 1. The Airy representation is not merely a formal change of variables, but the explicit representation of the eigenfunctions of a self-adjoint operator with compact resolvent.

3.3 Mixed boundary conditions and unified Airy quantization

In this section, we impose the mixed boundary condition at the origin, derive the corresponding quantization equation, and establish the basic spectral properties.

Theorem 4 (Unified Airy quantization law)**.** *Let $F > 0$, and consider the half-line problem (1) together with the decay condition $\psi \in L^2(0,\infty)$ and the mixed boundary condition*

$$a\psi(0) + b\psi'(0) = 0, \qquad (a,b) \neq (0,0). \tag{10}$$

Then every admissible eigenfunction is, up to a multiplicative constant, given by (9), and the corresponding eigenvalues are characterized by

$$a\mathrm{Ai}(-\lambda) + \frac{b}{\ell}\mathrm{Ai}'(-\lambda) = 0. \tag{11}$$

Equivalently, if $b \neq 0$ and

$$\rho := \frac{a\ell}{b},$$

then the spectrum is characterized by

$$\mathrm{Ai}'(-\lambda) + \rho\mathrm{Ai}(-\lambda) = 0. \tag{12}$$

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The corresponding energy levels are given by

$$E = F\ell\,\lambda. \tag{13}$$

Proof. By Section 3.2, every admissible solution is given by $\psi(x) = C\mathrm{Ai}(\frac{x}{\ell} - \lambda)$. Evaluating at $x = 0$, we obtain $\psi(0) = C\mathrm{Ai}(-\lambda)$ and $\psi'(0) = \frac{C}{\ell}\mathrm{Ai}'(-\lambda)$. Substituting these expressions into (10) yields $aC\mathrm{Ai}(-\lambda) + \frac{b}{\ell}C\mathrm{Ai}'(-\lambda) = 0$. Since $C \neq 0$ for a nontrivial eigenfunction, this is equivalent to (11). If $b \neq 0$, dividing by b/ℓ yields (12). Finally, (13) follows directly from the definition of λ in (8). □

Corollary 3 (Dirichlet and Neumann limits)**.** *The quantization law* (11) *includes the classical Dirichlet and Neumann cases:*

1. *If $b = 0$, then $\mathrm{Ai}(-\lambda) = 0$, yielding the Dirichlet spectrum.*
2. *If $a = 0$, then $\mathrm{Ai}'(-\lambda) = 0$, yielding the Neumann spectrum.*

Proposition 2 (Simplicity of the mixed spectrum)**.** *Every root of the quantization equation $a\mathrm{Ai}(-\lambda) + \frac{b}{\ell}\mathrm{Ai}'(-\lambda) = 0$, where $(a, b) \neq (0, 0)$, is simple.*

Proof. If $b = 0$, then $a \neq 0$, and the quantization equation reduces to $\mathrm{Ai}(-\lambda) = 0$. Since a nontrivial solution of the Airy equation cannot vanish together with its derivative at the same point, every zero of Ai is simple.

Assume now that $b \neq 0$, and define $\rho := a\ell/b$. The quantization equation becomes $g_\rho(\lambda) := \mathrm{Ai}'(-\lambda) + \rho\mathrm{Ai}(-\lambda) = 0$. Let λ_0 be a root. Then $\mathrm{Ai}'(-\lambda_0) = -\rho\mathrm{Ai}(-\lambda_0)$, and $\mathrm{Ai}(-\lambda_0) \neq 0$, since otherwise $\mathrm{Ai}(-\lambda_0) = \mathrm{Ai}'(-\lambda_0) = 0$, which contradicts Lemma 2. Using the Airy equation $\mathrm{Ai}''(t) = t\mathrm{Ai}(t)$, we obtain

$$\int_{-\lambda_0}^{\infty} \mathrm{Ai}(t)^2\,\mathrm{d}t = \mathrm{Ai}'(-\lambda_0)^2 + \lambda_0\mathrm{Ai}(-\lambda_0)^2 = (\lambda_0 + \rho^2)\mathrm{Ai}(-\lambda_0)^2 > 0.$$

Hence $\lambda_0 + \rho^2 > 0$. On the other hand, $g'_\rho(\lambda) = \lambda\mathrm{Ai}(-\lambda) - \rho\mathrm{Ai}'(-\lambda)$, and therefore $g'_\rho(\lambda_0) = (\lambda_0 + \rho^2)\mathrm{Ai}(-\lambda_0) \neq 0$. Thus every root of the quantization equation is simple. □

Proposition 3 (Positive roots and interlacing)**.** *Let $a_1 < a_2 < \cdots$ denote the positive zeros of $\mathrm{Ai}(-a_n) = 0$, and define $g_\rho(\lambda) = \mathrm{Ai}'(-\lambda) + \rho\mathrm{Ai}(-\lambda)$. Then, for each fixed $\rho \in \mathbb{R}$,*

1. *g_ρ has exactly one positive root in each interval (a_n, a_{n+1}), $n \geq 1$.*
2. *g_ρ has a positive root in $(0, a_1)$ if and only if $\rho < \rho_c := -\mathrm{Ai}'(0)/\mathrm{Ai}(0)$.*
3. *Consequently, the positive Robin roots interlace with the Dirichlet roots.*

Proof. On any interval $I \subset (0, \infty)$ that does not contain a zero of $\mathrm{Ai}(-\lambda)$, define $R(\lambda) := \mathrm{Ai}'(-\lambda)/\mathrm{Ai}(-\lambda)$. Using the quotient rule together with the Airy equation, we obtain

$$R'(\lambda) = \frac{-\mathrm{Ai}''(-\lambda)\mathrm{Ai}(-\lambda) + \mathrm{Ai}'(-\lambda)^2}{\mathrm{Ai}(-\lambda)^2} = \frac{\lambda\mathrm{Ai}(-\lambda)^2 + \mathrm{Ai}'(-\lambda)^2}{\mathrm{Ai}(-\lambda)^2} = \lambda + R(\lambda)^2.$$

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Since $\lambda > 0$ and $R(\lambda)^2 \geq 0$, it follows that $R'(\lambda) > 0$. Therefore, R is strictly increasing on every interval where it is defined, in particular on $(0, a_1)$.

Now fix $n \geq 1$. On the interval (a_n, a_{n+1}), the denominator $\mathrm{Ai}(-\lambda)$ has no zeros in the interior and simple zeros at the endpoints. Hence R is continuous and strictly increasing on (a_n, a_{n+1}), with vertical asymptotes at a_n and a_{n+1}. Moreover,

$$\lim_{\lambda \downarrow a_n} R(\lambda) = -\infty, \qquad \lim_{\lambda \uparrow a_{n+1}} R(\lambda) = +\infty.$$

Therefore, the equation $R(\lambda) = -\rho$ has exactly one solution in (a_n, a_{n+1}), proving assertion 1.

On the interval $(0, a_1)$, the same argument shows that R is continuous and strictly increasing, with $R(0) = \mathrm{Ai}'(0)/\mathrm{Ai}(0)$ and $\lim_{\lambda \uparrow a_1} R(\lambda) = +\infty$. Hence, the equation $R(\lambda) = -\rho$ has a solution in $(0, a_1)$ if and only if $-\rho > R(0)$, that is, $\rho < -\mathrm{Ai}'(0)/\mathrm{Ai}(0) = \rho_c$, establishing assertion 2.

Finally, assertion 3 follows immediately from assertion 1, since each interval (a_n, a_{n+1}) contains exactly one positive Robin root. □

Remark 7. Proposition 3 provides a complete characterization of the positive mixed spectrum and shows that the positivity of the spectrum is not guaranteed for an arbitrary Robin boundary parameter. Indeed, for $\lambda = -x < 0$, the quantization condition can be written as $\rho = h(x) := -\mathrm{Ai}'(x)/\mathrm{Ai}(x)$, $x > 0$.

The function $h(x)$ is strictly increasing on $[0, \infty)$, satisfies $h(0) = \rho_c$, and diverges to $+\infty$ as $x \to \infty$. Using the Airy equation $\mathrm{Ai}''(x) = x\mathrm{Ai}(x)$, one obtains

$$h'(x) = \frac{\mathrm{Ai}'(x)^2 - x\mathrm{Ai}(x)^2}{\mathrm{Ai}(x)^2} = \frac{\displaystyle\int_x^\infty \mathrm{Ai}(t)^2\,\mathrm{d}t}{\mathrm{Ai}(x)^2} > 0,$$

which proves the strict monotonicity of h.

Consequently, $\rho < \rho_c$ if and only if the spectrum is strictly positive, $\rho = \rho_c$ if and only if $E_0 = 0$, and $\rho > \rho_c$ if and only if the spectrum contains exactly one negative eigenvalue.

Figure 1 provides a geometric interpretation of the spectral transition induced by the Robin boundary condition at the critical value ρ_c. The figure shows that varying the boundary parameter continuously modifies the admissible quantization condition while the confining linear potential remains fixed. As a consequence, the lowest eigenvalue moves across the spectral threshold: for $\rho < \rho_c$ the spectrum is strictly positive, at $\rho = \rho_c$ a threshold state with zero energy appears, and for $\rho > \rho_c$ a unique negative bound state emerges. This behavior illustrates that the transition is entirely driven by the boundary interaction rather than by a change in the external potential, providing a clear physical interpretation of how boundary conditions alone can alter the spectral properties of a confined quantum system.

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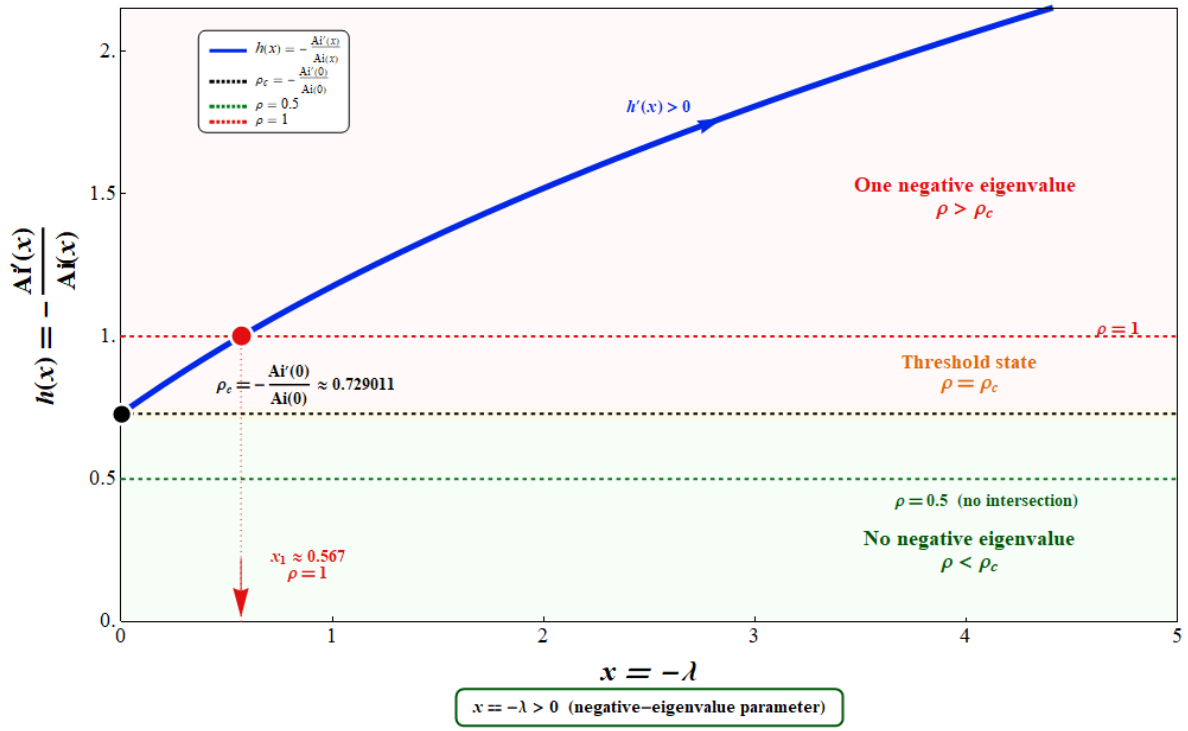


Figure 1. Geometric interpretation of the spectral transition generated by the Robin boundary parameter. The critical value $\rho_c = -\mathrm{Ai}'(0)/\mathrm{Ai}(0)$ separates three regimes: $\rho < \rho_c$ (strictly positive spectrum), $\rho = \rho_c$ (threshold state), and $\rho > \rho_c$ (exactly one negative bound state).

Corollary 4 (Weyl asymptotics of the mixed spectrum)**.** *Let $\lambda_n(\rho)$ denote the increasing sequence of positive roots of* $\mathrm{Ai}'(-\lambda) + \rho\mathrm{Ai}(-\lambda) = 0$. *Then*

$$\lambda_n(\rho) = \left(\frac{3\pi}{2}n\right)^{2/3} + O\left(n^{-1/3}\right), \qquad n \to \infty. \tag{14}$$

Proof. By Proposition 3, up to a shift of at most one index, each positive mixed root lies between two consecutive Dirichlet roots. More precisely, there exists $\kappa(\rho) \in \{0, 1\}$ such that $a_{n-\kappa(\rho)} < \lambda_n(\rho) < a_{n+1-\kappa(\rho)}$ for all sufficiently large n, where a_n is defined by $\mathrm{Ai}(-a_n) = 0$.

The classical asymptotic formula for the zeros of Ai is

$$a_n = \left(\frac{3\pi}{2}\left(n - \frac{1}{4}\right)\right)^{2/3} + O(n^{-4/3}), \qquad n \to \infty,$$

see [1, §9.9] or [2, Ch. 3]. Moreover,

$$\left(n - \frac{1}{4}\right)^{2/3} = n^{2/3}\left(1 - \frac{1}{4n}\right)^{2/3} = n^{2/3} + O(n^{-1/3}), \qquad n \to \infty.$$

Substituting this expansion into the asymptotic formula for a_n yields $a_n = \left(\frac{3\pi}{2}n\right)^{2/3} + O(n^{-1/3})$, and similarly $a_{n+1} = \left(\frac{3\pi}{2}n\right)^{2/3} + O(n^{-1/3})$. Since $\lambda_n(\rho)$ lies between two consecutive terms of this

asymptotic sequence, the estimate (14) follows immediately. □

Corollary 4 describes the asymptotic distribution of the positive Robin eigenvalues. Figure 2 illustrates the interlacing property established in Proposition 3, showing that each interval between consecutive Dirichlet zeros contains exactly one Robin eigenvalue, while an additional positive root appears only for $\rho < \rho_c$. The continuous displacement of the roots with ρ demonstrates that the boundary condition acts as a spectral control parameter without modifying the confining linear potential.

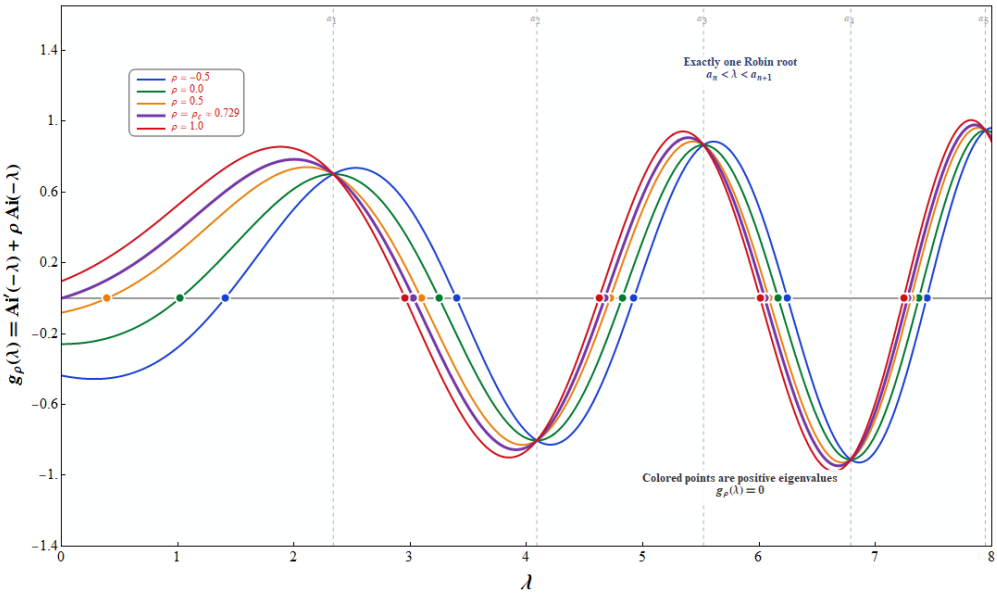


Figure 2. Quantization functions $g_\rho(\lambda) = \mathrm{Ai}'(-\lambda) + \rho\mathrm{Ai}(-\lambda)$ for representative Robin parameters. The intersections with the horizontal axis determine the positive eigenvalues, whereas the dashed vertical lines mark the Dirichlet eigenvalues a_n. The figure illustrates the interlacing property and the additional positive root that appears only when $\rho < \rho_c$.

Proposition 4 (Orthogonality)**.** *Let ψ_n and ψ_m be eigenfunctions corresponding to distinct eigenvalues $E_n \neq E_m$ of the mixed problem (1)–(10). Then*

$$\int_0^\infty \psi_n(x)\,\overline{\psi_m(x)}\,\mathrm{d}x = 0.$$

Proof. Applying Green's identity (5) to the differential expression

$$\tau u = -c\,u'' + F x\,u, \qquad c = \frac{\hbar^2}{2m},$$

with $u = \psi_n$ and $v = \psi_m$, we obtain

$$-c\Big[\psi_n'(x)\overline{\psi_m(x)} - \psi_n(x)\overline{\psi_m'(x)}\Big]_0^\infty = (E_n - E_m)\int_0^\infty \psi_n(x)\overline{\psi_m(x)}\,\mathrm{d}x.$$

The boundary contribution at infinity vanishes because ψ_n and ψ_m are proportional to the decaying

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Airy branch, and both Ai and Ai′ decay exponentially as $x \to +\infty$. The boundary contribution at the origin also vanishes because both eigenfunctions satisfy the same mixed boundary condition (10). Since $E_n \neq E_m$, it follows that

$$\int_0^\infty \psi_n(x)\overline{\psi_m(x)}\,\mathrm{d}x = 0.$$

□

Proposition 5 (Closed normalization formula)**.** *Assume that $b \neq 0$, and let λ_n be a root of (12). Then*

$$\int_{-\lambda_n}^\infty \mathrm{Ai}(t)^2\,\mathrm{d}t = (\lambda_n + \rho^2)\mathrm{Ai}(-\lambda_n)^2. \tag{15}$$

Consequently, a normalized eigenfunction is given by

$$\psi_n(x) = \frac{\mathrm{Ai}(x/\ell - \lambda_n)}{\sqrt{\ell(\lambda_n + \rho^2)}\,|\mathrm{Ai}(-\lambda_n)|}.$$

Proof. Using the Airy equation $\mathrm{Ai}''(t) = t\mathrm{Ai}(t)$, we obtain the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{Ai}'(t)^2 - t\mathrm{Ai}(t)^2\right) = -\mathrm{Ai}(t)^2. \tag{16}$$

Integrating over the interval $[-\lambda_n, \infty)$ yields

$$\int_{-\lambda_n}^\infty \mathrm{Ai}(t)^2\,\mathrm{d}t = \mathrm{Ai}'(-\lambda_n)^2 + \lambda_n\mathrm{Ai}(-\lambda_n)^2, \tag{17}$$

since both $\mathrm{Ai}(t)$ and $\mathrm{Ai}'(t)$ vanish as $t \to +\infty$. Since λ_n satisfies (12),

$$\mathrm{Ai}'(-\lambda_n) = -\rho\mathrm{Ai}(-\lambda_n),$$

substituting this identity into (17) gives (15). Finally,

$$1 = \int_0^\infty |\psi_n(x)|^2\,\mathrm{d}x = |C_n|^2\ell\int_{-\lambda_n}^\infty \mathrm{Ai}(t)^2\,\mathrm{d}t,$$

from which the normalized expression for ψ_n follows. □

Remark 8. Proposition 5 recovers the classical Dirichlet normalization formula

$$\int_{-a_n}^\infty \mathrm{Ai}(t)^2\,\mathrm{d}t = \mathrm{Ai}'(-a_n)^2,$$

when $\mathrm{Ai}(-a_n) = 0$, and the classical Neumann normalization formula

$$\int_{-a_n'}^\infty \mathrm{Ai}(t)^2\,\mathrm{d}t = a_n'\mathrm{Ai}(-a_n')^2,$$

when $\mathrm{Ai}'(-a_n') = 0$.

The explicit normalization formula makes it possible to compare the quantum states associated with different boundary conditions. Figure 3 displays the first three normalized Airy eigenfunctions for the Dirichlet and Robin realizations, illustrating how the boundary condition modifies the spatial localization, the position of the maxima, and the nodal structure of the bound states while the confining linear potential remains unchanged. In particular, the Robin boundary condition changes the behavior of the wave functions near the boundary, leading to a continuous redistribution of the probability density without altering the overall Airy character of the eigenfunctions.

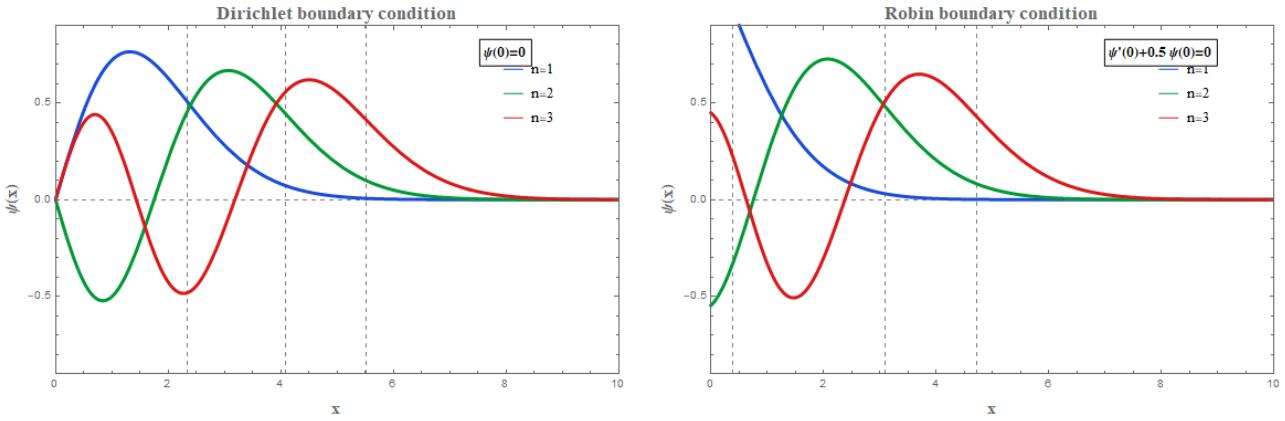


Figure 3. First three normalized Airy eigenfunctions for Dirichlet and Robin boundary conditions. Although the confining linear potential is identical in both cases, the Robin boundary condition shifts the maxima and modifies the spatial localization of the quantum states, while preserving their oscillatory Airy character.

Figure 3 illustrates the physical role of the boundary condition in shaping the quantum states. While the linear potential determines the overall Airy profile, the Robin boundary condition redistributes the probability density near the boundary, shifting the maxima of the eigenfunctions and modifying their localization without changing the confining potential itself. This shows that spectral and spatial properties can be controlled through the boundary interaction alone.

Illustrative spectral values

The analytical results obtained in the previous section provide a complete qualitative description of the mixed spectrum through the quantization law, the interlacing properties, and the asymptotic distribution of the positive eigenvalues. It is therefore natural to complement this analysis with representative numerical values illustrating the influence of the boundary parameter on the lowest energy levels. Since the low-lying states are the most sensitive to the boundary interaction, they provide the clearest picture of the spectral changes induced by the Robin condition.

Table 1 lists the first three *positive* dimensionless eigenvalues for two representative mixed boundary conditions. The case $\rho = 0$ corresponds to the Neumann boundary condition, whereas $\rho = 0.5$ represents a genuine Robin realization. Consistent with the interlacing and monotonicity properties established above, increasing the Robin parameter shifts the eigenvalues while preserving the overall ordering of the spectrum.

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Table 1. First three positive roots λ_n of $\mathrm{Ai}'(-\lambda) + \rho \mathrm{Ai}(-\lambda) = 0$.

ρ	λ_1	λ_2	λ_3
0	1.018793	3.248198	4.820099
0.5	0.392886	3.094540	4.717032

3.4 Perturbed linear potentials

We now consider a perturbation of the unperturbed operator by a small additional potential. Let

$$\widehat{H}_\varepsilon := \widehat{H}_0 + \varepsilon W(x), \qquad \widehat{H}_0 := -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + Fx,$$

on $(0,\infty)$, subject to the same mixed boundary condition (10).

Throughout this section, we assume that W is real-valued and belongs to $L^\infty(0,\infty)$. Under this assumption, Proposition 6 shows that $\{\widehat{H}_\varepsilon\}$ forms a holomorphic family of type (A) in the sense of Kato. Consequently, standard analytic perturbation theory applies, and the simple eigenvalues of $\widehat{H}_0$ admit first-order Rayleigh–Schrödinger expansions near $\varepsilon = 0$ [8].

Proposition 6 (Bounded perturbations and holomorphic family)**.** *Assume that $W \in L^\infty(0,\infty;\mathbb{R})$, and let M_W denote the multiplication operator by W on $L^2(0,\infty)$. Then M_W is bounded and, in particular, $\widehat{H}_0$-bounded with relative bound zero:*

$$\|M_W u\| \le \|W\|_{L^\infty}\|u\| = 0 \cdot \|\widehat{H}_0 u\| + \|W\|_{L^\infty}\|u\|, \qquad u \in D(\widehat{H}_0).$$

Consequently,

$$\widehat{H}_\varepsilon = \widehat{H}_0 + \varepsilon M_W, \qquad D(\widehat{H}_\varepsilon) = D(\widehat{H}_0),$$

is a holomorphic family of type (A) in the sense of Kato. In particular, every simple eigenvalue of $\widehat{H}_0$ admits a local analytic continuation in ε, and the corresponding first-order Rayleigh–Schrödinger formulas apply.

Proof. Since $W \in L^\infty(0,\infty)$, the multiplication operator M_W is bounded on $L^2(0,\infty)$, satisfying

$$\|M_W u\| \le \|W\|_{L^\infty}\|u\|, \qquad u \in L^2(0,\infty).$$

Hence M_W is $\widehat{H}_0$-bounded with relative bound zero. Therefore, the domain remains unchanged,

$$D(\widehat{H}_\varepsilon) = D(\widehat{H}_0),$$

and, for every $u \in D(\widehat{H}_0)$, the map

$$\varepsilon \longmapsto \widehat{H}_\varepsilon u = \widehat{H}_0 u + \varepsilon M_W u$$

is entire. Thus $\{\widehat{H}_\varepsilon\}$ is a holomorphic family of type (A) in the sense of Kato [8]. The analyticity of

the simple eigenvalues and their corresponding eigenvectors then follows from the standard analytic perturbation theory. □

Let $(E_n^{(0)}, \psi_n^{(0)})$ denote the normalized eigenpairs of $\widehat{H}_0$. By Theorem 4,

$$E_n^{(0)} = F\ell\,\lambda_n, \qquad \psi_n^{(0)}(x) = C_n^{(0)}\mathrm{Ai}\left(\frac{x}{\ell} - \lambda_n\right), \tag{18}$$

where λ_n is a solution of (11).

We seek asymptotic expansions of the form

$$E_n(\varepsilon) = E_n^{(0)} + \varepsilon E_n^{(1)} + \mathcal{O}(\varepsilon^2), \qquad \psi_n(\varepsilon) = \psi_n^{(0)} + \varepsilon\chi_n + \mathcal{O}(\varepsilon^2), \tag{19}$$

subject to the orthogonality condition

$$\langle \psi_n^{(0)}, \chi_n\rangle = 0. \tag{20}$$

Theorem 5 (First-order spectral correction)**.** *Under the assumptions of Proposition 6, the first-order correction to the n-th eigenvalue is*

$$E_n^{(1)} = \langle \psi_n^{(0)}, W\psi_n^{(0)}\rangle = \int_0^\infty W(x)|\psi_n^{(0)}(x)|^2\,\mathrm{d}x. \tag{21}$$

Moreover, the first-order correction to the corresponding eigenfunction is

$$\chi_n = \sum_{m\neq n} \frac{\langle \psi_m^{(0)}, W\psi_n^{(0)}\rangle}{E_n^{(0)} - E_m^{(0)}}\,\psi_m^{(0)}. \tag{22}$$

Proof. Substituting (19) into the eigenvalue equation $\widehat{H}_\varepsilon\psi_n(\varepsilon) = E_n(\varepsilon)\psi_n(\varepsilon)$ and comparing equal powers of ε, the zeroth-order terms yield $\widehat{H}_0\psi_n^{(0)} = E_n^{(0)}\psi_n^{(0)}$, whereas the first-order terms give

$$(\widehat{H}_0 - E_n^{(0)})\chi_n = -(W - E_n^{(1)})\psi_n^{(0)}. \tag{23}$$

Taking the inner product of (23) with $\psi_n^{(0)}$ and using the self-adjointness of $\widehat{H}_0$, we obtain

$$\langle \psi_n^{(0)}, (\widehat{H}_0 - E_n^{(0)})\chi_n\rangle = \langle (\widehat{H}_0 - E_n^{(0)})\psi_n^{(0)}, \chi_n\rangle = 0.$$

Hence, $0 = -\langle \psi_n^{(0)}, W\psi_n^{(0)}\rangle + E_n^{(1)}\langle \psi_n^{(0)}, \psi_n^{(0)}\rangle$. Since $\|\psi_n^{(0)}\| = 1$, equation (21) follows.

By Theorem 3, the eigenfunctions of $\widehat{H}_0$ form an orthonormal basis of $L^2(0,\infty)$. Therefore, together with the gauge condition (20), we may write $\chi_n = \sum_{m\neq n} c_{m,n}\psi_m^{(0)}$.

Taking the inner product of (23) with $\psi_m^{(0)}$, where $m \neq n$, we obtain $\langle \psi_m^{(0)}, (\widehat{H}_0 - E_n^{(0)})\chi_n\rangle = -\langle \psi_m^{(0)}, W\psi_n^{(0)}\rangle$, since $\langle \psi_m^{(0)}, \psi_n^{(0)}\rangle = 0$. Because $\widehat{H}_0\psi_m^{(0)} = E_m^{(0)}\psi_m^{(0)}$, it follows that $(E_m^{(0)} - E_n^{(0)})c_{m,n} = -\langle \psi_m^{(0)}, W\psi_n^{(0)}\rangle$. Hence

$$c_{m,n} = \frac{\langle \psi_m^{(0)}, W\psi_n^{(0)}\rangle}{E_n^{(0)} - E_m^{(0)}},$$

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which proves (22).

To justify the convergence of the series in (22), note that $W \in L^\infty(0,\infty)$ implies $W\psi_n^{(0)} \in L^2(0,\infty)$. Moreover, Proposition 1 shows that $\widehat{H}_0$ has compact resolvent and hence purely discrete spectrum, while Proposition 2 establishes that the quantization equation has only simple roots. Therefore, each eigenvalue $E_n^{(0)}$ is simple and isolated, so the spectral gap

$$\delta_n := \operatorname{dist}\Big(E_n^{(0)}, \sigma(\widehat{H}_0) \setminus \{E_n^{(0)}\}\Big)$$

is strictly positive. Hence, Parseval's identity yields

$$\sum_{m\neq n} \left| \frac{\langle \psi_m^{(0)}, W\psi_n^{(0)}\rangle}{E_n^{(0)} - E_m^{(0)}} \right|^2 \leq \frac{1}{\delta_n^2} \sum_{m\neq n} |\langle \psi_m^{(0)}, W\psi_n^{(0)}\rangle|^2 \leq \frac{\|W\psi_n^{(0)}\|^2}{\delta_n^2} < \infty.$$

Hence the series in (22) converges in $L^2(0,\infty)$. □

Formula (21) admits a convenient representation in Airy coordinates. Using (18) together with the change of variables $t = x/\ell - \lambda_n$, we obtain

$$\begin{aligned} E_n^{(1)} &= |C_n^{(0)}|^2 \int_0^\infty W(x) \mathrm{Ai}\Big(\frac{x}{\ell} - \lambda_n\Big)^2 \,\mathrm{d}x \\ &= |C_n^{(0)}|^2 \ell \int_{-\lambda_n}^\infty W(\ell(t+\lambda_n)) \mathrm{Ai}(t)^2 \,\mathrm{d}t. \end{aligned} \tag{24}$$

On the other hand, $1 = |C_n^{(0)}|^2 \ell \int_{-\lambda_n}^\infty \mathrm{Ai}(t)^2 \,\mathrm{d}t$. Eliminating the normalization constant $C_n^{(0)}$, we obtain

$$E_n^{(1)} = \frac{\displaystyle\int_{-\lambda_n}^\infty W(\ell(t+\lambda_n)) \mathrm{Ai}(t)^2 \,\mathrm{d}t}{\displaystyle\int_{-\lambda_n}^\infty \mathrm{Ai}(t)^2 \,\mathrm{d}t}. \tag{25}$$

If $b \neq 0$, Proposition 5 yields the explicit expression

$$E_n^{(1)} = \frac{\displaystyle\int_{-\lambda_n}^\infty W(\ell(t+\lambda_n)) \mathrm{Ai}(t)^2 \,\mathrm{d}t}{(\lambda_n + \rho^2)\mathrm{Ai}(-\lambda_n)^2}.$$

Remark 9. Formula (25) shows that the first-order correction is an Airy-weighted average of the perturbing potential. Consequently, it is completely determined by the unperturbed Airy profile and the corresponding quantization parameter λ_n.

Example 1 (Exponentially decaying bounded perturbation). Consider the bounded perturbation $W(x) = e^{-x}$. Since $W \in L^\infty(0,\infty)$ and $0 < W(x) \leq 1$, the assumptions of Proposition 6 are satisfied.

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Formula (25) yields

$$E_n^{(1)} = \frac{\displaystyle\int_{-\lambda_n}^{\infty} e^{-\ell(t+\lambda_n)}\mathrm{Ai}(t)^2\,\mathrm{d}t}{\displaystyle\int_{-\lambda_n}^{\infty}\mathrm{Ai}(t)^2\,\mathrm{d}t}.$$

If $b \neq 0$, Proposition 5 provides the equivalent expression

$$E_n^{(1)} = \frac{\displaystyle\int_{-\lambda_n}^{\infty} e^{-\ell(t+\lambda_n)}\mathrm{Ai}(t)^2\,\mathrm{d}t}{(\lambda_n+\rho^2)\mathrm{Ai}(-\lambda_n)^2}.$$

Moreover, since $0 < e^{-x} \leq 1$, it follows immediately that $0 < E_n^{(1)} \leq 1$. Thus, the perturbation produces a positive first-order correction whose magnitude is governed by the localization of the corresponding Airy eigenstate near the origin.

Example 2 (Rationally decaying bounded perturbation). Consider the bounded perturbation $W(x) = \frac{1}{1+x^2}$. Again, $W \in L^\infty(0,\infty)$ and $0 < W(x) \leq 1$, so the assumptions of Proposition 6 are satisfied. The first-order correction is given by

$$E_n^{(1)} = \frac{\displaystyle\int_{-\lambda_n}^{\infty} \frac{\mathrm{Ai}(t)^2}{1+\ell^2(t+\lambda_n)^2}\,\mathrm{d}t}{\displaystyle\int_{-\lambda_n}^{\infty}\mathrm{Ai}(t)^2\,\mathrm{d}t}.$$

If $b \neq 0$, this expression becomes

$$E_n^{(1)} = \frac{\displaystyle\int_{-\lambda_n}^{\infty} \frac{\mathrm{Ai}(t)^2}{1+\ell^2(t+\lambda_n)^2}\,\mathrm{d}t}{(\lambda_n+\rho^2)\mathrm{Ai}(-\lambda_n)^2}.$$

As in the previous example, $0 < E_n^{(1)} \leq 1$. Unlike the exponential case, however, the perturbation decays more slowly, so the first-order correction samples a broader portion of the Airy profile.

Remark 10. Examples 1 and 2 illustrate two physically distinct bounded perturbations of the linear confining potential. The exponential profile $W(x) = e^{-x}$ is localized near the boundary and therefore primarily affects the low-lying states, whereas the rational profile $W(x) = 1/(1+x^2)$ decays more slowly and influences a broader spatial region, leading to the larger first-order corrections reported in Tables 2 and 3. Since both perturbations are positive, they produce positive first-order corrections to the energy levels. Moreover, because both perturbations are bounded, they preserve the confining character of the operator and hence the discreteness of the spectrum, as the dominant behavior at infinity remains governed by the linear potential $Fx \to +\infty$.

The combined effect of the perturbing potential and the Robin boundary parameter is summarized in Figure 4. The comparison highlights how different perturbation profiles modify the low-lying energy levels while preserving the qualitative behavior of the spectrum.

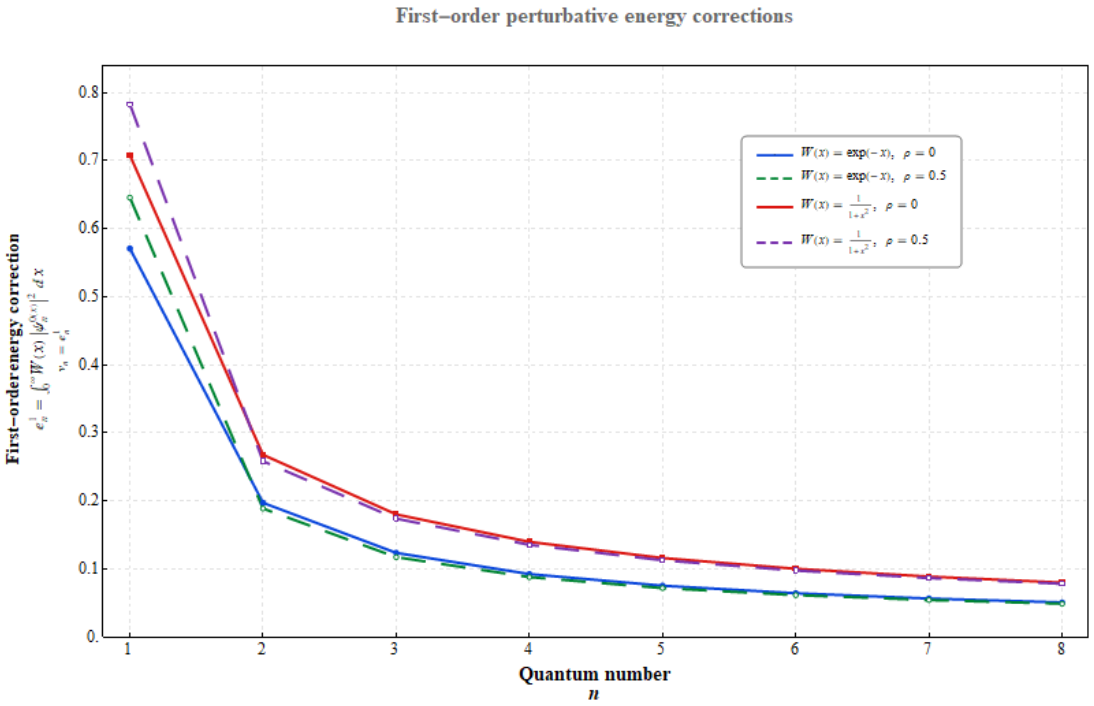


Figure 4. First-order energy corrections $E_n^{(1)} = \int_0^\infty W(x)|\psi_n(x)|^2\,\mathrm{d}x$ for the exponential and rational perturbations under Neumann ($\rho = 0$) and Robin ($\rho = 0.5$) boundary conditions. The corrections decrease with the quantum number, showing that highly excited states are less sensitive to localized perturbations. The rational potential produces systematically larger shifts than the exponential one, while the Robin boundary condition modifies the response in a state-dependent manner, increasing the ground-state correction and slightly reducing the next two corrections shown.

The corresponding coefficients are reported below for the same boundary parameters.

Illustrative bounded perturbative coefficients

For the exponentially decaying perturbation $W(x) = e^{-x}$, define

$$\nu_n^{(\exp)} := E_n^{(1)} = \frac{\displaystyle\int_{-\lambda_n}^{\infty} e^{-\ell(t+\lambda_n)}\,\mathrm{Ai}(t)^2\,\mathrm{d}t}{\displaystyle\int_{-\lambda_n}^{\infty} \mathrm{Ai}(t)^2\,\mathrm{d}t}.$$

Similarly, for the rational perturbation $W(x) = 1/(1+x^2)$, define

$$\nu_n^{(\mathrm{rat})} := E_n^{(1)} = \frac{\displaystyle\int_{-\lambda_n}^{\infty} \frac{\mathrm{Ai}(t)^2}{1+\ell^2(t+\lambda_n)^2}\,\mathrm{d}t}{\displaystyle\int_{-\lambda_n}^{\infty} \mathrm{Ai}(t)^2\,\mathrm{d}t}.$$

These coefficients can be computed numerically for any prescribed value of the boundary parameter ρ. In both cases, they represent positive bounded first-order corrections satisfying $0 < \nu_n^{(\exp)}, \nu_n^{(\mathrm{rat})} \leq 1$.

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The first three coefficients for the exponentially and rationally decaying perturbations are reported in Tables 2 and 3, respectively. As expected from its slower spatial decay, the rational perturbation produces consistently larger first-order corrections.

Table 2. Dimensionless first-order coefficients $\nu_n^{(\mathrm{exp})}$ for the bounded perturbation $W(x) = e^{-x}$ with $\ell = 1$.

ρ	$\nu_1^{(\mathrm{exp})}$	$\nu_2^{(\mathrm{exp})}$	$\nu_3^{(\mathrm{exp})}$
0	0.570929	0.197408	0.123873
0.5	0.645695	0.189005	0.117640

Table 3. Dimensionless first-order coefficients $\nu_n^{(\mathrm{rat})}$ for the bounded perturbation $W(x) = 1/(1+x^2)$ with $\ell = 1$.

ρ	$\nu_1^{(\mathrm{rat})}$	$\nu_2^{(\mathrm{rat})}$	$\nu_3^{(\mathrm{rat})}$
0	0.707932	0.267922	0.180398
0.5	0.782767	0.258549	0.174147

Remark 11. The roots λ_n used to compute the coefficients reported in Tables 2 and 3 were obtained numerically by solving the quantization equation $\mathrm{Ai}'(-\lambda) + \rho \mathrm{Ai}(-\lambda) = 0$ with a bracketing root-finding method on the intervals guaranteed by Proposition 3.

4. DISCUSSION

The results show that the half-line Schrödinger operator with a linear potential and mixed boundary conditions fits naturally within the framework of self-adjoint spectral theory while retaining the explicit analytical structure of the Airy equation. From the operator-theoretic viewpoint, Proposition 1 establishes a self-adjoint realization that is bounded from below and has compact resolvent. Consequently, the model possesses a purely discrete spectrum and a complete set of bound states, providing a rigorous spectral realization rather than a formal boundary-value problem.

Within this framework, the Airy reduction remains exact. The quantization theory developed in Section 3.3 shows that Dirichlet, Neumann, and Robin boundary conditions are all described through the same Airy functions, so the boundary interaction modifies the admissible eigenvalues without changing the underlying differential equation. The simplicity of the spectrum, orthogonality of the eigenfunctions, interlacing of the positive roots, and explicit normalization formulas provide a complete spectral characterization of the model.

The graphical results complement this analytical description with a physical interpretation. The quantization functions show that the Robin parameter continuously shifts the admissible energy levels, while the threshold analysis identifies the critical value separating a strictly positive spectrum from the appearance of a unique negative bound state. Likewise, Figure 3 shows that the linear potential determines the overall Airy profile, whereas the boundary condition modifies the localization of the quantum states by shifting the maxima of the eigenfunctions and redistributing the probability density

near the boundary. Thus, different self-adjoint realizations generate different bound-state configurations under the same confining potential.

The threshold at ρ_c has a direct boundary-interaction interpretation. In the quadratic form, the Robin condition contributes $-(\hbar^2/2m)(\rho/\ell)|\psi(0)|^2$; hence increasing ρ strengthens an attractive endpoint interaction and lowers every eigenvalue branch, consistently with $\mathrm{d}\lambda/\mathrm{d}\rho = -1/(\lambda+\rho^2) < 0$. At $\rho = \rho_c$ the ground state crosses zero, and for $\rho > \rho_c$ this boundary attraction produces the unique negative eigenvalue. This crossing does not signal a loss of confinement: the resolvent remains compact and the spectrum remains discrete.

The perturbative analysis in Section 3.4 preserves the same Airy framework. As summarized in Remark 10, $E_n^{(1)}$ measures the Airy-weighted spatial overlap of the perturbation with the unperturbed state: the exponential profile is localized near the boundary, whereas the more slowly decaying rational profile samples a broader region and produces larger shifts. Figure 4 and Tables 2–3 further show that the Robin response is state dependent rather than a uniform amplification of all levels. Since both profiles are bounded, they leave the dominant confining behavior $Fx \to +\infty$ and the spectral discreteness unchanged.

The comparison with Toloza and Uribe [14] is complementary. Their work provides sharp large-index asymptotics for eigenvalues and logarithmic norming constants under weighted regularity and decay assumptions, whereas Corollary 4 and Section 3.4 combine exact unperturbed quantization with fixed-level weak-coupling corrections $q = \varepsilon W$ for bounded W. Thus, their emphasis is high-energy spectral data, while the present analysis also resolves the transition at ρ_c and the state-dependent response of the low-lying levels.

Overall, the present work provides a unified operator-theoretic and spectral description of the half-line Airy problem in which boundary conditions and bounded perturbations are treated within the same analytical framework. From the physical viewpoint, the results show that boundary interactions influence both the energy spectrum and the spatial localization of the bound states while preserving the fundamental Airy structure generated by the confining linear potential.

5. CONCLUSIONS

This paper has developed a self-adjoint spectral framework for half-line Schrödinger operators with a linear potential and mixed boundary conditions in the Airy setting. We proved that the corresponding realization is bounded from below and has compact resolvent, implying a purely discrete spectrum. We also established a unified Airy quantization law for Dirichlet, Neumann, and Robin boundary conditions, together with the simplicity of the spectrum, orthogonality of eigenfunctions, interlacing of the corresponding roots, and explicit normalization formulas.

From a physical perspective, the results show that the confining linear potential determines the global Airy structure of the quantum states, whereas the boundary condition governs the selection of the admissible bound states and influences their spatial localization near the endpoint. The normalized eigenfunctions illustrate this effect explicitly while remaining within the same Airy framework.

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For bounded perturbations of the form $\widehat{H}_0 + \varepsilon W$, we derived first-order corrections to both eigenvalues and eigenfunctions in terms of explicit Airy-function integrals. This shows that the Airy representation remains analytically effective under small bounded perturbations of the potential, providing a consistent extension of the unperturbed spectral theory.

Overall, the results establish a unified framework connecting self-adjoint operator theory, Airy special functions, and one-dimensional quantum models with boundary interactions. The approach also provides a natural basis for future extensions to more general confining potentials, interface conditions, and higher-order perturbative analyses.

6. DECLARATION

Authors' contributions

Juan Toribio Milane contributed to the conceptualization of the study, the formal spectral analysis, the physical interpretation of the perturbed models, and the preparation of the original manuscript. José Angel Gómez Hernández contributed to the conceptualization of the study, the development of the spectral formulation under mixed boundary conditions, the Airy reduction, the formal spectral analysis, and the preparation of the original manuscript. Ángel M. Polanco Fernández contributed to the validation of the results, the analytical verification of the proofs, and the critical revision of the manuscript. All authors read and approved the final version of the manuscript.

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Competing interests

The authors declare that they have no competing interests.

Generative AI use

No generative AI tools were used in the preparation of this manuscript.

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